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A Converse to the Jordan Curve Theorem for Digital Curves[1]

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It is shown that a subset S of a digital picture is a simple closed curve if and only if its complement $\bar{S}$ has exactly two components, and every point of S is adjacent to both of these components.

The well-known Jordan curve theorem states that in the Euclidean plane, the complement of a simple closed curve C has exactly two connected components, one inside C and the other outside, and C is the common boundary of these two components. Fáry and Isenberg (1974) prove a converse to this theorem: If C is a set whose complement has exactly two components, and if each point of C is an accessible boundary point of both components, then C is a simple closed curve.

The purpose of this section is to establish a corresponding result for digital pictures. Specifically, we shall prove

THEOREM 1. *Let S be a subset of a digital picture such that*

(a) *$\bar{S}$ (the complement of S) has exactly two 8-components D, E*

(b) *Every point of S is 8-adjacent to both D and E.*

Then S is a simple 4-curve (having at least eight points).

For the definitions used here see Rosenfeld (1970, 1973, 1974).

We first need

PROPOSITION 2. *Let T be any subset of a digital picture; let U, V be 4-components of T, and let P, Q be 8-components of its complement $\bar{T}$. Then it is impossible that U and V are each adjacent to both P and Q.*

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Proof. By (Rosenfeld, 1974), the 4-components of T and 8-components of $\bar{T}$, under the relation "is adjacent to," form an acyclic graph. If U were adjacent to P, P to V, V to Q, and Q to U, this graph would have a cycle. ∎

COROLLARY 3. *If S is as in Theorem* 1, *it is* 4-*connected.*

Proof. Suppose S had two 4-components A, B; then each of them would be adjacent to both D and E, contradicting Proposition 2. ∎

Note. In Proposition 2 and Corollary 3, "adjacent" can mean either 4- or 8-, since if a component of 1's and a component of 0's are 8-adjacent, they must also be 4-adjacent.

PROPOSITION 4. *If S is as in Theorem* 1, *it contains a simple* 4-*curve C having more than four points.*

Proof. By Corollary 3, S is 4-connected; since $\bar{S}$ has exactly two 8-components, S has exactly one 8-hole. Hence, by (Rosenfeld, 1973, Corollary 4.5), S can be "shrunk" down to a simple 4-curve as required. ∎

We can now complete the proof of Theorem 1. Let D', E', be the two 8-components of $\bar{C}$. Then $D \subseteq D'$ and $E \subseteq E'$ (say); see (Rosenfeld, 1974, Proposition 1). Let x be a point of S that is not in C; thus, x is in D' or E', say the former. Since x is in S, there exist points d, e of D, E, respectively, that are 8-adjacent to x. But then $x \in D'$ is 8-adjacent to $e \in E \subseteq E'$, contradiction. Thus, C must be all of S, so that S is a simple 4-curve. ∎

The converse of Theorem 1 states that if S is a simple 4-curve having more than four points, its complement has exactly two 8-components, and every point of S is 8-adjacent to both of these components. This is an immediate restatement of Proposition 3.3 and Theorem 3.4 of Rosenfeld (1973).

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